Math 4200
Friday November 6

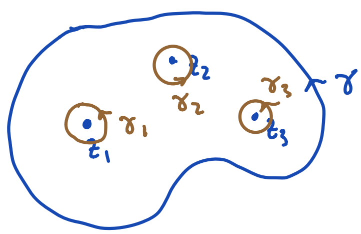
4.1 Calculating residues at isolated singularities
4.2 The Residue Theorem(s) - tie together and extend previous contour integration tricks.

Announcements:

Residue Theorem (Replacement Theorem version): Let f be analytic on a region A, except on a finite set of isolated singularities $\{z_1, z_2, \ldots z_k\} \subseteq A$ Let γ be a simple closed contour in A which contains none of the singularities, and which bounds a a subregion B containing some of the singularities, in the counterclockwise direction. Then

$$\int_{\gamma} f(z)\, dz = 2\,\pi\, i \sum_{z_j \in B} Res(f, z_j).$$

proof: Use The section 2.2 Replacment Theorem for domains with holes together Laurent series for f at the singularities, and the diagram below. Notice our notation for residues...



Exercise: Show - just for the practice - that this theorem includes as special cases

(a) Cauchy's theorem that $\int_{\gamma} f(z)\, dz = 0$ if f is analytic in A.

(b) CIF $f(z_0) = \dfrac{1}{2\,\pi\, i} \int_{\gamma} \dfrac{f(z)}{z - z_0}\, dz$ if z_0 is inside γ and f is analytic in A.

So we would like systematic ways to compute residues. There's a page in the book...

Prop 4.1.7. $f(z) = \frac{g(z)}{h(z)}$ $g(z_0) \neq 0$. $h(z)$ zero of order $k-1$, i.e. $h^{(k)}(z)$ is 1st non-zero deriv.

the residue at $z_0, \operatorname{Res}(g/h; z_0)$ *is given by*

$$\operatorname{Res}(g/h; z_0) = \left[\frac{k!}{h^{(k)}(z_0)}\right]^k \times$$

$$\begin{vmatrix} \frac{h^{(k)}(z_0)}{k!} & 0 & 0 & \cdots & 0 & g(z_0) \\ \frac{h^{(k+1)}(z_0)}{(k+1)!} & \frac{h^{(k)}(z_0)}{k!} & 0 & \cdots & 0 & g^{(1)}(z_0) \\ \frac{h^{(k+2)}(z_0)}{(k+2)!} & \frac{h^{(k+1)}(z_0)}{(k+1)!} & \frac{h^{(k)}(z_0)}{k!} & \cdots & 0 & \frac{g^{(2)}(z_0)}{2!} \\ \vdots & \vdots & \vdots & & & \vdots \\ \frac{h^{(2k-1)}(z_0)}{(2k-1)!} & \frac{h^{(2k-2)}(z_0)}{(2k-2)!} & \frac{h^{(2k-3)}(z_0)}{(2k-3)!} & \cdots & \frac{h^{(k+1)}(z_0)}{(k+1)!} & \frac{g^{(k-1)}(z_0)}{(k-1)!} \end{vmatrix},$$

where the vertical bars denote the determinant of the enclosed $k \times k$ *matrix.*

Table 4.1.1 Techniques for Finding Residues

In this table g and h are analytic at z_0 and f has an isolated singularity. The most useful and common tests are indicated by an asterisk.

Function		Test	Type of Singularity	Residue at z_0
1.	$f(z)$	$\lim_{z\to z_0}(z-z_0)f(z) = 0$	removable	0
*2.	$\frac{g(z)}{h(z)}$	g and h have zeros of same order	removable	0
*3.	$f(z)$	$\lim_{z\to z_0}(z-z_0)f(z) = 0$ exists and is $\neq 0$	simple pole	$\lim_{z\to z_0}(z-z_0)f(z)$
*4.	$\frac{g(z)}{h(z)}$	$g(z_0)\neq 0, h(z_0) = 0,$ $h'(z_0)\neq 0$	simple pole	$\frac{g(z_0)}{h'(z_0)}$
5.	$\frac{g(z)}{h(z)}$	g has zero of order k, h has zero of order $k+1$	simple pole	$(k+1)\frac{g^{(k)}(z_0)}{h^{(k+1)}(z_0)}$
*6.	$\frac{g(z)}{h(z)}$	$g(z_0)\neq 0$ $h(z_0) = 0 = h'(z_0)$ $h''(z_0)\neq 0$	second-order pole	$2\frac{g'(z_0)}{h''(z_0)} - \frac{2}{3}\frac{g(z_0)h'''(z_0)}{[h''(z_0)]^2}$
*7.	$\frac{g(z)}{(z-z_0)^2}$	$g(z_0)\neq 0$	second-order pole	$g'(z_0)$
*8.	$\frac{g(z)}{h(z)}$	$g(z_0) = 0, g'(z_0)\neq 0,$ $h(z_0) = 0 = h'(z_0)$ $= h''(z_0), h'''(z_0)\neq 0$	second-order pole	$3\frac{g''(z_0)}{h'''(z_0)} - \frac{3}{2}\frac{g'(z_0)h^{(iv)}(z_0)}{[h'''(z_0)]^2}$
9.	$f(z)$	k is the smallest integer such that $\lim_{z\to z_0}\phi(z_0)$ exists where $\phi(z) = (z-z_0)^k f(z)$	pole of order k	$\lim_{z\to z_0}\frac{\phi^{(k-1)}(z)}{(k-1)!}$
*10.	$\frac{g(z)}{h(z)}$	g has zero of order l, h has zero of order $k+l$	pole of order k	$\lim_{z\to z_0}\frac{\phi^{(k-1)}(z)}{(k-1)!}$ where $\phi(z) = (z-z_0)^k\frac{g}{h}$
11.	$\frac{g(z)}{h(z)}$	$g(z_0)\neq 0, h(z_0) =$ $\ldots = h^{k-1}(z_0)$ $= 0, h^k(z_0)\neq 0$	pole of order k	see Proposition 4.1.7.

Table entry 4: Let $f(z) = \dfrac{g(z)}{h(z)}$ where $g(z_0) \neq 0$, $h(z_0) = 0$, $h'(z_0) \neq 0$. Prove that f has a pole of order 1, and

$$Res(f, z_0) = \frac{g(z_0)}{h'(z_0)}$$

Example: The table entry above is great for quotient functions in general, as long as $h(z)$ only has zeroes of order 1. So it makes many rational function contour integrals a breeze, without having to use long division and/or partial fractions.

Compute $\displaystyle\int_{|z|=3} \frac{z^3}{z^2 - 2z} dz$ (And notice that it would be just as easy for an arbitrary analytic function in the numerator.

Table entry 7: If $f(z)$ has a pole of order k at z_0, and has the form

$$f(z) = \frac{g(z)}{\left(z - z_0\right)^k}$$

then

$$Res\left(f, z_0\right) = \frac{g^{(k-1)}\left(z_0\right)}{(k-1)!}$$

Example Compute $Res\left(\frac{e^{2z}}{(z-1)^2}; 1\right)$

Table entry 6: Let $f(z) = \frac{g(z)}{h(z)}$ where $g(z_0) \neq 0$, $h(z_0) = h'(z_0) = 0$, $h''(z_0) \neq 0$.

Then f has a pole of order 2 and

$$Res(f, z_0) = \frac{2\, g'(z_0)}{h''(z_0)} - \frac{2}{3}\frac{g(z_0)h'''(z_0)}{h''(z_0)^2} \qquad !!!$$

(This leads into Prop 4.1.7, your extra credit hw problem.)

Residue Theorem (Deformation Theorem version, more general than the Green's Theorem version.). Let Let f be analytic on a region A, except on a finite set of isolated singularities $\{z_1, z_2, \ldots z_k\} \subseteq A$. Let γ be a closed curve which is homotopic to a point in A. Then

$$\int_\gamma f(z)\, dz = 2\pi i \sum_{j=1}^{k} Res(f, z_j)\, I(\gamma; z_j)$$

Being more general than the Green's Theorem version, this proof is also a bit more complicated. For each isolated singularity z_j we have the Laurent series

$$S_j(z) = S_{j1}(z) + S_{j2}(z) = \sum_{n=0}^{\infty} a_{jn}(z - z_j)^n + \sum_{m=1}^{\infty} \frac{b_{jm}}{(z - z_j)^m}.$$

Because the z_j are point singularities, the singular part of the series, $S_{j2}(z)$ converges in $\mathbb{C} \setminus \{z_j\}$, and the non-singular part converges for $0 \leq |z - z_j| < R_j$ for some positive radius of convergence R_j.

Now consider

$$g(z) := f(z) - \sum_{j=1}^{k} S_{j2}(z).$$

Explain why $g(z)$ has removable singularities at each z_l:

Thus we may consider g to be analytic in A, so since γ is homotopic as closed curves to a point in A,

$$\int_{\gamma} g(z)\, dz = 0.$$

Expand this to get the result!

Math 4200-001
Homework 11
4.1-4.2
Due Wednesday November 11 at 11:59 p.m.
Exam will cover thru 4.2

4.1 1de, 3, 5, 7ab, 9
4.2 2 (Section 2.3 Cauchy's Theorem), 3, 4, 6, 9, 13.

w11.1 (extra credit) Prove Prop 4.1.7, the determinant computation for the residue at an order k pole for $f(z) = \frac{g(z)}{h(z)}$ at z_0, where $g(z_0) \neq 0$. (Hint: it's Cramer's rule for a system of equations.)